Parabolic and Borel subgroups

Jack Petok

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G is a linear algebraic group over k, k algebraically closed.

Let us recall a few facts about complete varieties. The proofs are relatively easy, we refer the reader to the text. We make heavy use of these facts in proving the main theorems.

Definition 1. A variety X is said to be *complete* if, for all varietes X, the projection $X \times Y \to Y$ is a closed mapping.

Lemma 1. *Let X be a complete variety.*

(i) Any closed subvariety of Y is complete.

(ii) If Y is also complete, then $X \times Y$ is complete.

(iii) If $\varphi\colon X \to Y$ is a morphism of varieties, then $\varphi(X)$ is closed and complete subvariety of Y.

(iv) If $X \subset Y$, then X is closed.

(v) If X is irreducible, then the regular functions on X are all constant.

(vi) If X is affine, then X is finite.

Definition 2. A closed subgroup $P \subseteq G$ is a *parabolic subgroup* if G/P is a complete variety.

Recall that for a closed subgroup H, G/H is a quasi-projective variety of dimension $\dim G - \dim H$, i.e. is an open subvariety of $\mathbf{P}^n$. Since complete subvarieties are closed, this proves

Lemma 2. *(6.2.2) G/P is a projective variety.*

Parabolicity is transitive, in the following sense:

Lemma 3. *Let P be parabolic in G, Q parabolic in P. Then Q is parabolic in G.*

Proof. We need to show $\pi\colon G/Q \times X \to X$ is a closed mapping for any variety X. To prove this, it suffices to show that , for closed $A \subset G \times X$ such that

$$(g, x) \in A \iff (gQ, x) \subseteq A$$

we have that $\pi(A)$ is closed in X. Define

$$\alpha\colon P \times G \times X \to G \times X$$

$$\alpha(p, g, x) = (gp, x).$$

Then the set

$$A' = \alpha^{-1}(A) = \{(p, g, x) \mid (gp, x) \in A\} = \{(p, gp^{-1}, x) \mid (g, x) \in A, p \in P\}$$

is closed in $P \times G \times X$. Note that

$$(p, g, x) \in A' \iff (gp, x) \in A \iff (gpQ, x) \subset A \iff (pQ, g, x) \subset A'$$

Thus, setting

$$\varphi \colon P \times G \times X \to P/Q \times G \times X$$
$$(p, g, x) \mapsto (pQ, g, x),$$

we observe that $\varphi(A'^c) = \varphi(A')^c$. Since φ is an open mapping (5.3.2(i)), this observation proves that $\varphi(A')$ is closed. Let C denote the projection of $\varphi(A')$ in $G \times X$. Since P/Q is complete, the C is closed. Note that we can also write

$$C = \cup_{(g,x) \in A}(gP, x).$$

Thus, we see that

$$(g', x) \in C \iff (g'P, x) \subseteq C$$

By an analogous argument to the one in the previous paragraph, the projection of C in $G/P \times X$ is closed, call this closed set $C' \subseteq G/P \times X$. Since G/P is complete, it follows that the projection of C' to X is closed, but this projection is exactly $\pi(A)$. $\pi(A)$ is thus closed, which is what we wanted to prove. □

As a partial converse, subgroups containing parabolic subgroups are also parabolic, as the lemma first part of the following lemma shows.

Lemma 4. *(6.2.4)*

(i) Let P be a parabolic subgroup of G. If Q is a closed subgroup of G containing P, then Q is parabolic in G.

(ii) P is parabolic in $G \iff P^0$ is parabolic in G^0.

Proof. G/Q is complete since it is the image of the surjection $G/P \to G/Q$, proving (i).

To prove (ii), note that G^0 is parabolic in G and P^0 is parabolic in P (remember G/G^0 is just a finite set, so is a dimension zero projective variety). First suppose P is parabolic in G. Then P^0 is parabolic in G by the previous lemma, so G/P^0 is projective. Since G^0/P^0 is a closed subvariety of G/P^0, it is also complete, so P^0 is parabolic in G^0. Conversely, if P^0 is parabolic in G^0, then P_0 is parabolic in G by the previous lemma, so P is parabolic in G by (i). □

Lemma 5. *(6.2.1) If there is a bijective morphism of G-spaces $X \to Y$, then X is complete if and only if Y is complete.*

Proof. embedded in proof of next proposition □

Proposition 1. *A connected LAG G contains proper parabolic subgroups if and only if G is non-solvable.*

Proof. First, suppose G has no proper parabolic subgroups. As usual, we can assume G is a closed subgroup of some $GL(V)$, $V \simeq k^n$. G then naturally acts on $\mathbf{P}(V)$. Let X be a closed orbit for this action, which we know exists since closed orbits exist for G-spaces. Then X is a complete, projective variety. For $x \in X$, we have that G_x, the isotropy group for x. The orbit-stabilizer theorem gives a bijective morphism of homogenous spaces:

$$G/P \to X$$

$$gG_x \mapsto g.x.$$

For any variety Y, then, we get a bijective morphism of G-spaces

$$G/G_x \times Y \to X \times Y$$

$$(gG_x, y) \to (g.x, y)$$

which is an open mapping by 5.3.2, and thus a homeomorphism. Thus, the projection $G/G_x \times Y \to Y$ is a closed mapping, because $X \times Y \to Y$ is closed mapping (as X is complete). So G_x is parabolic. Write $P_0 = G_x$. In the case that $G_x = G$, set $V_1 = V/x$ (i.e. mod out by a 1-D subspace, since x is a line in V). Then G acts on $\mathbf{P}(V_1)$. Take a closed orbit, and get a parabolic subgroup P_1 as before, by looking at some isotropy. If $P_1 = G$, mod out by another 1-D subspace corresponding to a point in the isotropy. Eventually, we will get a proper parabolic subgroup, or we will exhaust dimensions. If we exhaust all dimensions, then, geometrically, every element of G fixes a chain of n subspaces, one in each dimension. Thus, in a certain basis, every element of G is upper triangular. Thus, G is solvable (just start taking commutators of group of upper triangular matrix, and the first commutator makes all diagonal entries 1, and each successive kills a super diagonal, making all entries in that superdiagonal equal to 0).

Conversely, suppose G is solvable. We show G has no parabolic subgroup via induction on $\dim G$ Suppose for the sake of contradiction that G has a proper parabolic subgroup. Let P be a proper parabolic subgroup of maximal dimension. Since $G = G^0$, it suffices to show that the closed connected group P^0 is parabolic in G in light of the last lemma. $[G, G]$ is a closed, connected subgroup of G. Let $Q = P^0.[G, G]$. Q is a group becasue $[G, G]$ is normal in G. Then Q is a closed, connected group of G containing P^0, so is parabolic. So either $Q = P^0$ or $Q = G$. If $Q = G$, then $G = P^0 \cdot [G, G]$, so the second isomorphism theorem says that the natural G-morphism

$$[G, G]/[G, G] \cap P^0 \to G/P^0$$

is bijective. Thus, $[G, G]/[G, G] \cap P^0$ is complete, so $[G, G] \cap P^0$ is parabolic in $[G, G]$. $[G, G]$ is of strictly smaller dimension than G, since G is solvable. By our induction hypothesis, $[G, G] \cap P^0 = [G, G]$, so $[G, G] \subseteq P^0$, which contradicts $Q = G$. If, on the other hand, $Q = P^0$, then $[G, G] \subseteq P^0$, so $P^0 \trianglelefteq G$ (basic group theory). Thus, from 5.5.10, G/P^0 is affine, but also complete, which means G/P^0 is finite and connected, so $G = P^0$. Contradiction. □

The following theorem is due to Armand Borel, and is known as Borel's Fixed Point Theorem.

Theorem 1. *Let G be a connected solvable LAG, and X a complete G-variety. Then there is some $x \in X$ such that $G_x = G$.*

Proof. There exists a closed orbit for G in X. For some x in the orbit, G_x is parabolic (as we showed in the proof of he previous proposition). But we must have $G_x = G$, by the previous proposition. □

Definition 3. A *Borel subgroup* of G is a closed, connected, solvable subgroup of G, which is maximal for these properties.

Borel subgroups clearly exist. Just pick a closed, connected, solvable subgroup of maximal dimension.

The next theorem gives an alternate definition of a Borel subgroup: B is Borel if and only if it a minimal parabolic subgroup.

Theorem 2. *(i) A closed subgroup of G is parabolic if and only if it contains a Borel subgroup.*

(ii) A Borel subgroup is parabolic.

(iii) Any two Borel subgroups are conjugate.

Proof. Any Borel subgroup is contained in G^0, and P is parabolic in G if and only if P^0 is parabolic in G^0. Thus, it suffices to assume G is connected.

First we prove the "only if" direction for (i). Let B be a Borel subgroup, P a parabolic subgroup. The Borel fixed point theorem applied to the complete B-variety G/P, there is some $gP \in G/P$ such that $B_{gP} = B$, i.e. $BgP \subset gP$, which shows that P contains a conjugate of a Borel subgroup, which is also Borel.

We now prove (ii). Assume G is non-solvable, or else (ii) is trivial as the only Borel subgroup is G itself, which is clearly parabolic. Then G has a proper parabolic subgroup P. We already showed that $B \subset P$ for some Borel subgroup B of G. Clearly, B is also Borel in P. Induction on dimension of G shows that B is parabolic in P. Thus, by transitivity of parabolicity, B is parabolic in G.

The "if" direction is now done, using an earlier lemma.

Let B, B' be two Borel subgroups. Then B' is conjugate to a subgroup of B and B is conjugate to a sbugroup of B'. Thus, $\dim B = \dim B'$, and we're done. □

Corollary 1. *(6.2.8) $\varphi\colon G \to G'$ surjective homomorphims of LAGs. Let P be a parabolic (resp. Borel) subgroup of G. Then φP is a subgroup of G' of the same type.*

Proof. By first part of theorem above, it suffices to consider the case of Borel subgroup P. Then $\varphi P$is closed, connected, solvable. Since $G/P \to G'/\varphi P$ is surjective, $G'/\varphi P$ is complete, so φP is parabolic, and so φP contains a Borel subgroup of G'. Thus, φP is Borel. □

A few remarks about the center of a Borel group. Here B is a Borel group of G.

Corollary 2. *G connected, then $Z(G)^0 \subset Z(B) \subset Z(G)$*

Proof. $Z(G)^0$ is closed and commutative (i.e. solvable) and connected, so is inside some Borel subgroup. Therefore, for some $g \in G$, $gZ(G)^0g^{-1} = Z(G)^0 \subseteq B$, hence the first inclusion.

For the other inclusion, if $g \in Z(B)$, the morphism $x \mapsto gxg^{-1}x^{-1}$ induces a morphism $G/B \to G$, since if $x = yb$, then $gyg^{-1}y^{-1} = gxb^{-1}g^{-1}bx^{-1} = gxg^{-1}x^{-1}$. Now G/B is complete, so the image in G is also complete, closed subvariety of G, i.e. a complete affine variety (since G is affine). So the image of G/B must be a point, e. Thus, $xg = gx$ for all $x \in G$, so $g \in Z(G)$, hence the second inclusion. □